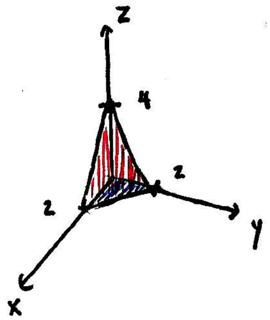
Q104 (Starting 2020)

Triple Integrals (Q201 Lesson 1) – C3211

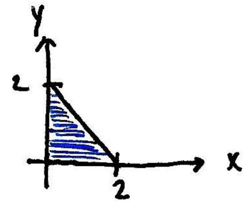
12. $\iiint_E y \, dV$ with E bounded by planes $x=0$, $y=0$, $z=0$, $2x+2y+z=4$



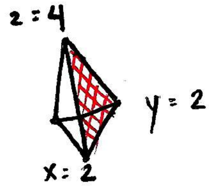
Note that: $dV = dx\,dy\,dz$

→ can integrate in any order

↓ projection



(base of the region)



Bottom: $z = 0$

Top: $z = 4 - 2x - 2y$

→ Integrate over the region from the plane $z = 0$ to $z = 4 - 2x - 2y$:

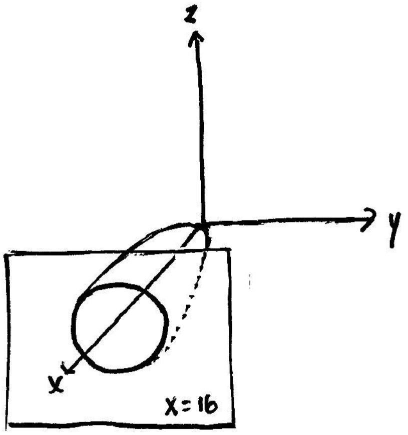
$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y \, dz\, dy\, dx = \int_0^2 \int_0^{2-y} \int_0^{4-2x-2y} y \, dz\, dx\, dy$$

Integrating up to surface

same integration as if only across a 2D region

can switch orders, just remember to change bounds

22. Solid enclosed by paraboloid $x = y^2 + z^2$ and plane $x = 16$

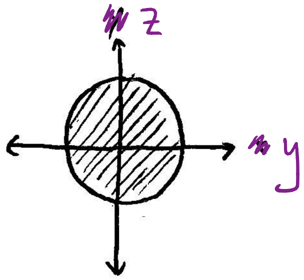


$$\boxed{\text{Volume} = \iiint_E dV}$$

$$V = \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{y^2+z^2}^{16} dx\, dz\, dy$$

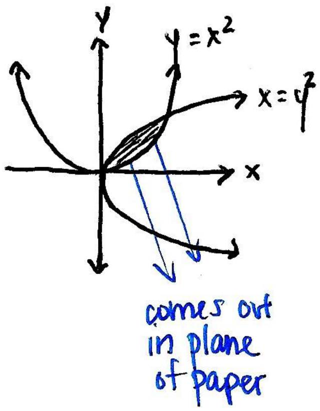
↓ polar transformation

T: $y = r\cos\theta$ $z = r\sin\theta$ $|J| = r$



$$V = \int_0^{2\pi} \int_0^{4} \int_{r^2}^{16} r\, dx\, dr\, d\theta$$

14. $\iiint_E xy\, dV$ bounded by cylinders $y = x^2$ and $x = y^2$ and planes $z = 0$ and $z = x + y$

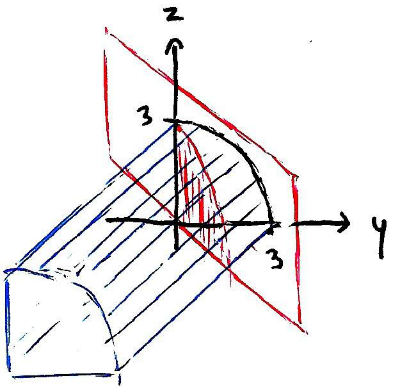


$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\, dz\, dy\, dx$$

from plane $z = 0$ to $z = x + y$

Fix just as if a 2D surface

18. $\iiint_E z\,dV$ bounded by cylinder $z^2+y^2=9$ and planes $x=0$, $y=3x$, $z=0$ (first octant)



Note the first octant only!

$$\iiint_E z\,dV = \int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{y/3} z\,dx\,dz\,dy = \int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx$$

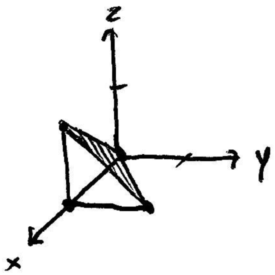
coming out in x, so put inside

Or with a polar transformation:

T: $y = r\cos\theta$ $z = r\sin\theta$ $|J| = r$

$$\iiint_E z\,dV = \int\int\int_0^1 (r\sin\theta)\,r\,dx\,dr\,d\theta$$

16. $\iiint_T xyz\,dV$ where T is solid tetrahedron with vertices (0,0,0) (1,0,0) (1,1,0) (1,0,1)



$$\int_0^1\int_0^x\int_0^{x-y} xyz\,dz\,dy\,dx$$

$x-y$ → see below (plane equation)

Need to get equation of plane:

$\vec{v} = \langle 1,1,0\rangle$ $\vec{u} = \langle 1,0,1\rangle$

$$\vec{v}\times\vec{u} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \vec{i} - \vec{j} - \vec{k}$$

$1(x) - 1(y) - 1(z) = 0$

$z = x - y$

Transformations in Triple Integrals (Q201 Lesson 2) — CA212

Key: $$\iiint_E f(x,y,z)\,dV = \iiint_T f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

$T:\ x = x(u,v,w) \quad y = y(u,v,w) \quad z = z(u,v,w)$

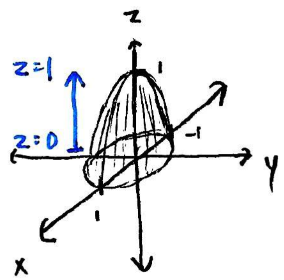
$$|J| = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

I. Rectangular to Cylindrical — $T:\ x = r\cos\theta \quad y = r\sin\theta \quad z = z \rightarrow \boxed{|J| = r}$

II. Rectangular to Spherical — $T:\ x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$

$\hookrightarrow \boxed{|J| = \rho^2\sin\phi}$

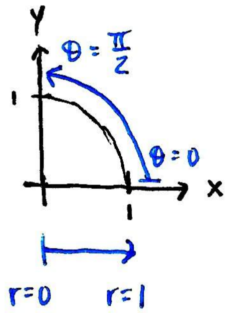
15.7 #18 $\iiint_E (x^3 + xy^2)\,dV$ where E is solid in first octant that lies beneath $z = 1 - x^2 - y^2$



$T:\ x = r\cos\theta$

$y = r\sin\theta$

$z = z$



$$\iiint_E (x^3 + xy^2)\,dV$$

$$= \int_0^{\pi/2}\int_0^{1}\int_0^{1-r^2} x(x^2+y^2)\,dV = \int_0^{\pi/2}\int_0^{1}\int_0^{1-r^2} r\cos\theta \cdot r^2 \cdot r\,dz\,dr\,d\theta$$

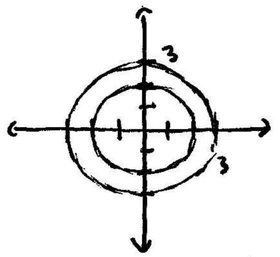
(θ r z)

$$= \int_0^{\pi/2}\int_0^{1}\int_0^{1-r^2} r^4\cos\theta\,dz\,dr\,d\theta$$

(θ r z)

15.7 #20 $\iiint_E x\,dV$ where E is enclosed by $z=0$ and $z=x+y+5$ and by $x^2+y^2=4$ and $x^2+y^2=9$

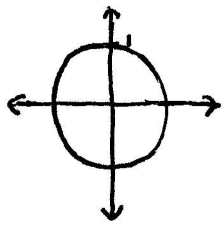
should be 2π



$$\iiint_E x\,dV = \int_0^{\pi/2}\int_2^3\int_0^{(r\cos\theta + r\sin\theta+5)} (r\cos\theta)\, r\,dz\,dr\,d\theta$$

15.7 #22 Volume of solid lying within $x^2+y^2=1$ and sphere $x^2+y^2+z^2=4$

(can with caps)



$$V = \int_0^{2\pi}\int_0^1\int_{(-\sqrt{4-r^2})}^{(\sqrt{4-r^2})} r\,dz\,dr\,d\theta$$

15.8 #20 (Shape is essentially sphere shell in octants 2,3,4)

$$V = \int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 f(x(\rho,\theta,\phi), y(\dots), z(\dots))\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

ϕ θ ρ — Gen. function — Jacobian

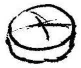
15.8 #22 $\iiint_H (9-x^2-y^2)\,dV$ where H is solid hemisphere $x^2+y^2+z^2\le 9$, $z\ge 0$

$$V = \int_0^3\int_0^{2\pi}\int_0^{\pi/2}\left(9-(\rho\cos\theta\sin\phi)^2-(\rho\sin\theta\sin\phi)^2\right)\rho^2\sin\theta\,d\phi\,d\theta\,d\rho$$

ρ θ ϕ

Jacobian should be $\rho^2\sin\phi$

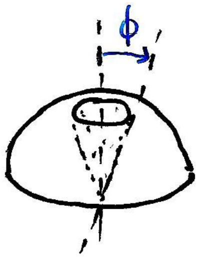
15.8 #26 $\iiint_E xyz\, dV$ where E lies between spheres $\rho = 2$ and $\rho = 3$ ← should be $\rho = 4$
and above cone $\phi = \pi/3$



(like a pumpkin top)

$$= \int_0^{\pi/3}\int_0^{2\pi}\int_2^{4} (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\sin\phi) \cdot \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

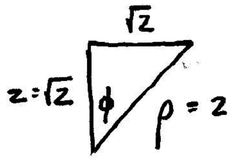
15.8 #30 Volume of solid within sphere $x^2+y^2+z^2=4$, above $z=0$, below cone $z=\sqrt{x^2+y^2}$



Need to figure out this angle ϕ:

$z=\sqrt{x^2+y^2} \rightarrow z^2 = x^2+y^2$

$x^2+y^2+z^2 = 4 \longrightarrow z = \sqrt{2}$



$\tan\phi = 1 \rightarrow \phi = \pi/4$

$$\Rightarrow V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{2} \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

→ Proof of Cylindrical Jacobian :

$T: \; x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \frac{\partial x}{\partial r}\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} - \frac{\partial x}{\partial \theta}\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial z} \end{vmatrix} + \frac{\partial x}{\partial z}\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta - 0) + r\sin\theta\,(\sin\theta - 0) + 0$$

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r \qquad |J| = r \quad \text{QED.}$$

→ Proof of spherical Jacobian :

$T: \; x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \frac{\partial z}{\partial \rho}\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} - \frac{\partial z}{\partial \theta}\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{vmatrix} + \frac{\partial z}{\partial \phi}\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \cos\phi\,(\underbrace{\rho\sin\phi\cdot -\sin\theta}_{\partial x/\partial\theta}\cdot\underbrace{\rho\sin\theta\cos\phi}_{\partial y/\partial\phi} - \underbrace{\rho\sin\phi\cos\theta}_{\partial y/\partial\theta}\cdot\underbrace{\rho\cos\theta\cos\phi}_{\partial x/\partial\phi}) - 0$$

$$+ \rho\cdot -\sin\phi\,(\cos\theta\sin\phi\cdot\rho\sin\phi\cos\theta - \sin\theta\sin\phi\cdot\rho\sin\phi\cdot -\sin\theta$$

$$= \cos\phi\,(-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi) \rightarrow -\rho^2\sin\phi\cos^2\phi$$

$$-\rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \rightarrow -\rho^2\sin^3\phi$$

$$= -\rho^2\sin\phi\,(\cos^2\phi + \sin^2\phi) = -\rho^2\sin\phi \qquad |J| = \rho^2\sin\phi \quad \text{QED.}$$